HYDRONAUTICS, Incorporated

TECHNICAL REPORT 7103-2

# HYDRONAUTICS, incorporated
## research in hydrodynamics

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A STUDY OF THE NONLINEAR WAVE RESISTANCE OF A TWO-DIMENSIONAL SOURCE GENERATED BODY

By

Gedeon Dagan

February 1972

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Prepared for

Office of Naval Research
Department of the Navy
under
Contract No. N00014-71-C-0080
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# NOTATION

| | | |
|---|---|---|
| $a^I, a^{II}$ | - | amplitude of free waves |
| $A^s, B^s, C^s_{jk}, E^s_{jk}, K^s_{jk}, G^s$ | - | coefficients of the free-waves resulting from the second order free-surface correction |
| $B^b, E^b_{jk}, I^b_{jk}$ | - | as above for the body correction |
| $C = 0.5722 \ldots$ | - | Euler constant |
| $C_D = D'/0.5\rho U'^2 T'$ | - | drag coefficient |
| $C_L = D'/\rho U'^2 L'$ | - | drag coefficient |
| $D'$ | - | wave resistance ($D = D'g/\rho U'^4$) |
| $D^I$ | - | first order wave resistance |
| $D^{II}$ | - | second order wave resistance |
| $D^{IIs}, D^{IIb}$ | - | wave resistance resulting from second order free-surface and body effects, respectively |
| $\overline{Ei}(i\zeta)$ | - | exponential integral |
| $f'$ | - | complex potential ($f = f'g/U'^3$) |
| $f^I, f^{II}$ | - | first and second order complex potentials, respectively |
| $g$ | - | acceleration of gravity |
| $h'$ | - | depth of submersion ($h = h'g/U'^2$) |
| $\ell'_{jk}$ | - | distance between the sources j and k ($\ell_{jk} = \ell'_{jk} g/U'^2$) |
| $L'$ | - | forebody or body length ($L = L'g/U'^2$) |
| $p^{II}$ | - | second order pressure |
| $q'_j$ | - | strength of source j |
| $T'$ | - | thickness |
| $T'_j$ | - | thickness change at source j |

| | | |
|---|---|---|
| $t = t'/h'$ | - | thickness (dimensionless with respect to h') |
| $x', y'$ | - | cartesian coordinates ($x, y = x'g/U'^2,\ y'g/U'^2$) |
| $z' = x' + iy'$ | - | complex variable ($z = z'g/U'^2$) |
| $u', v'$ | - | velocity components ($u = u'/U'$, $v = v'/U'$) |
| $U'$ | - | unperturbed velocity |
| $w' = u' - iv'$ | - | complex velocity ($w = w'/U'$) |
| $\alpha$ | - | real part of $\omega$ |
| $\beta$ | - | imaginary part of $\omega$ |
| $\gamma_\ell$ | - | auxiliary function |
| $\delta_\ell$ | - | auxiliary function |
| $\epsilon_j = 2\pi q_j' g/U'^3$ | - | strength of source j (dimensionless) |
| $\epsilon_L = T'/L'$ | - | slenderness parameter |
| $\phi'$ | - | potential ($\phi = \phi' g/U'^3$) |
| $\chi_\ell$ | - | auxiliary function |
| $\psi'$ | - | stream function ($\psi = \psi' g/U'^3$) |
| $\zeta$ | - | complex variable |
| $\eta'$ | - | free-surface elevation ($\eta = \eta' g/U'^2$) |
| $\omega(\zeta) = e^{-i\zeta}\bar{E}i(i\zeta)$ | - | function associated with waves |
| $\lambda, \rho$ | - | complex variables |

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13. ABSTRACT


The second order wave resistance of an arbitrary distribution of sources, representing a thin symmetrical two-dimensional body, is derived in a closed analytical form.

The result is applied to a few shapes: an isolated source representing a semi-infinite blunt body, an open body with a leading edge of a fine shape and a closed body generated by a source and a sink. In each case the first order wave resistance as well as the body and free-surface second order corrections are derived separately.

The relevance of the results to ship wave resistance is discussed in qualitative terms.


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| Free-surface Correction | | | | | | |
| Body Correction | | | | | | |
| Source Generated Body | | | | | | |
| Source-sink Body | | | | | | |

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The relevance of the results to ship wave resistance is discussed in qualitative terms.

# I. INTRODUCTION

The wave resistance of bodies moving beneath or at a free-surface is generally computed by linearizing the equations of flow, i.e. by assuming that the body is thin or slender. This way the free-surface condition is linearized and the body is replaced by a distribution of singularities of known strength.

Till recently only the first order wave resistance has been computed in most cases. Nonlinear effects associated with the body condition have been investigated by Havelock (1926) and Giesing and Smith (1971) for two-dimensional flows and by Gadd (1970) and others for three-dimensional flows past ships. In all these works the free-surface condition has been kept in its first order version. The procedure is not consistent in principle because body and free-surface effects are of the same order of magnitude, in an asymptotic sense at least.

A complete second order numerical computation of the wave resistance has been carried out by Tuck (1965) in the case of a submerged cylinder. The free-surface correction turned out to be larger than the body correction in this case. Dagan (1971) has shown that as the cylinder approaches the free-surface, nonlinear effects have to be incorporated already in the first order approximation. Salveson (1969) has determined numerically the second order wave resistance of a submerged symmetrical hydrofoil. The body correction appeared to be the largest in this case, apparently because of the important role played by the circulation associated with the Kutta-Juvkovsky condition at the trailing edge.

The computation of the nonlinear wave resistance for two-dimensional bodies is of a limited interest in applications. In fact, the main purpose of the works mentioned so far was to draw conclusions on ship wave resistance from the two-dimensional results, at least on a qualitative basis. The two-dimensional solution is considerably simpler than that of three-dimensional flow and it is worthwhile, therefore, to explore first the two-dimensional case in order to acquire insight in the problem and experience in handling the mathematical difficulties.

The present work, dealing with the wave resistance of a two-dimensional source generated body, is a continuation of Tuck and Salvesen works, but differ from them in: (i) the solution is carried out for bodies of arbitrary thickness distribution rather than particular cases, (ii) the second order wave resistance is obtained in a closed analytical form, rather than by tedious numerical computations, such that a general analysis of the results is possible, and (iii) the wave resistance is computed for two-dimensional bodies resembling ships, i.e. elongated bodies of a fine form and without circulation. The cylinder has an extremely blunt shape and in the case of the hydrofoil the contribution of the circulation has no three-dimensional counterpart.

## II. EQUATIONS AND BOUNDARY CONDITIONS

We consider a discrete distribution of sources of strength $q_j'$ (Fig. ), located at $x' = \xi_j'$, $y' = 0$ $(j=1,2,\ldots,n)$. The sources are assumed to be at the same elevation because any thin body without circulation degenerates into such a distribution. It is quite easy, however, to extend the approach presented here to any distribution of sources or other singularities.

We make the variables dimensionless as follows

$$x = x'g/U'^2, \quad y = y'g/U'^2, \quad z = x+iy = z'g/U'^2, \quad \eta = \eta'g/U'^2$$

$$w = u-iv = w'/U', \quad f = \phi + i\psi = f'g/U'^3, \quad h = h'g/U'^2, \tag{1}$$

$$\epsilon_j = 2\pi q_j' g/U'^3$$

where $f'$ is the complex potential, $w'$ is the complex velocity, $h'$ the submergeance depth, $U'$ the unperturbed velocity and $\eta'$ the free-surface elevation (Fig. 1).

If we relate $q_j'$ to the change in the thickness $T_j'$ of a body moving in an infinite flow domain, then

$$\epsilon_j = \frac{T_j' g}{U'^2} \quad . \tag{2}$$

$f$ and $w$ are now expanded in asymptotic series

$$f = z + f^{I} + f^{II} + \ldots \tag{3}$$

$$w = \frac{df}{dz} = 1 + w^{I} + w^{II} + \ldots \tag{4}$$

where $f^{I}, w^{I} = O(\epsilon)$ and $f^{II}, w^{II} = O(\epsilon^2)$, these being the only terms considered herein.

The analytical functions $f^{I}(z)$ and $f^{II}(z)$ satisfy the following linearized free-surface conditions (see, for instance, Wehausen and Laitone, 1960)

$$\mathrm{Re}(f^{I}_{,z} + if^{I}) = p^{I}(x) = 0 \qquad (y = h) \tag{5}$$

$$\mathrm{Re}(f^{II}_{,z} + if^{II}) = p^{II}(x) = \frac{3}{2}(u^{I})^2 + \frac{1}{2}(v^{I})^2 \qquad (y = h) \tag{6}$$

and the radiation condition

$$f_{I} \to 0 , \qquad f_{II} \to 0 \qquad (x \to -\infty) \tag{7}$$

The free-surface elevation $\eta = \eta^{I} + \eta^{II} + \ldots$ is given by

$$\eta^{I}(x) = -\psi^{I}(x,h) \tag{8}$$

$$\eta^{II}(x) = \frac{\eta^{I}(x)^2}{2} - \psi^{II}(x,h) \quad . \tag{9}$$

The body condition reduces to the requirement that *at any order* the thickness change at the source $j$ is equal to $T_j'$. At second order we, therefore, stipulate

$$f = z + \frac{\varepsilon_j}{2\pi}\,\ell n(z-x_j) + O(\varepsilon^3) \qquad |z - x_j| = \varepsilon_j \tag{10}$$

The purpose of the following sections is to determine the wave resistance of the source distribution.

## III. THE FREE WAVES BEHIND A SOURCE DISTRIBUTION

To compute the wave resistance we need only the expression of the free waves trailing far behind the body. Moreover, the solution for a pair of sources may be easily extended to an arbitrary number of singularities. For the sake of simplicity we carry out first the analysis for two sources, say of strength $\varepsilon_j$ and $\varepsilon_k$, located at $x_j$ and $x_k$, respectively.

### 1. First order solution

The first order solution is well known (Wehausen and Laitone, 1960) and is given here for convenience of reference

$$f_{jk}^{I} = \frac{\varepsilon_j}{2\pi}\,\ell n(z-x_j) + \frac{\varepsilon_k}{2\pi}\,\ell n(z-x_k) + \frac{\varepsilon_j}{2\pi}\,\ell n(z-x_j-2ih) + \frac{\varepsilon_k}{2\pi}\,\ell n(z-x_k-2ih) - \frac{\varepsilon_j}{2\pi}\,\omega(z-x_j-2ih) - \frac{\varepsilon_k}{2\pi}\,\omega(z-x_k-2ih) \tag{11}$$

$$w^I_{ij} = \frac{\varepsilon_j}{2\pi}\frac{1}{z-x_j} + \frac{\varepsilon_k}{2\pi}\frac{1}{z-x_k} - \frac{\varepsilon_j}{2\pi}\frac{1}{z-x_j-2ih} - \frac{\varepsilon_k}{2\pi}\frac{1}{z-x_k-2ih}$$
$$+ \frac{i\varepsilon_j}{\pi}\,\omega(z-x_j-2ih) + \frac{i\varepsilon_k}{2\pi}\,\omega(z-x_k-2ih) \qquad (12)$$

The function $\omega(\zeta)$ defined as

$$\omega(\zeta) = \alpha + i\beta = e^{-i\zeta}\overline{Ei}(i\zeta) = e^{-i\zeta}\int_{-\infty}^{\zeta}\frac{e^{i\lambda}}{\lambda}\,d\lambda \qquad (13)$$

represents the trailing singularities associated with the free waves. The properties of $\omega(\zeta)$, which will be extensively used in the sequel, are discussed in Appendix.

The far downstream potential is obtained by expanding (11) (see Eq. (65))

$$f^I_{jk} = -2i\varepsilon_j\, e^{-i(z-x_j-2ih)} - 2i\varepsilon_k\, e^{-i(z-x_k-2ih)} \qquad (x \to \infty) \qquad (14)$$

and the free waves profile (by Eq. (8)) is given by

$$\eta^I_{jk}(x) = -\mathrm{Im}[(-2i)e^{-h}(\varepsilon_j e^{ix_j} + \varepsilon_k e^{ix_k})e^{-ix}] =$$

$$= 2e^{-h}[(\varepsilon_j\cos x_j+\varepsilon_k\cos x_k)\cos x + (\varepsilon_j\sin x_j+\varepsilon_k\sin x_k)\sin x]$$
$$(x \to \infty) \qquad (15)$$

2. Second order body correction

The velocity induced at $z = x_j$ by the various terms of (12), excepting the first term representing the source $j$ itself, is

$$w_j = \left(w^I_{jk} - \frac{\varepsilon_j}{2\pi}\frac{1}{z-x_j}\right)_{z=x_j} = \frac{\varepsilon_j}{2\pi}\frac{1}{2ih} + \frac{\varepsilon_k}{2\pi}\frac{1}{x_j-x_k} - \frac{\varepsilon_k}{2\pi}\frac{1}{x_j-x_k-2ih}$$
$$+ \frac{\varepsilon_j}{\pi}\,\omega(-2ih) + \frac{i\varepsilon_k}{\pi}\,\omega(x_j-x_k-2ih) \qquad (16)$$

We assume in the sequel that $x_j$ is an increasing sequence and that $k > j$. With $\ell_{jk} = x_k - x_j > 0$ $w_j$ becomes

$$w_j = \frac{\varepsilon_j}{2\pi}\frac{1}{2ih} - \frac{\varepsilon_k}{2\pi}\frac{1}{\ell_{jk}} + \frac{\varepsilon_k}{2\pi}\frac{1}{\ell_{jk}+2ih} + \frac{i\varepsilon_j}{\pi}\omega(-2ih) + \frac{i\varepsilon_k}{\pi}\omega(-\ell_{jk}-2ih) \quad (17)$$

By the same token, the velocity induced at $z = x_j$ is

$$w_k = \frac{\varepsilon_k}{2\pi}\frac{1}{2ih} + \frac{\varepsilon_k}{2\pi}\frac{1}{\ell_{jk}} - \frac{\varepsilon_j}{2\pi}\frac{1}{\ell_{jk}-2ih} + \frac{i\varepsilon_k}{\pi}\omega(-2ih) + \frac{i\varepsilon_j}{\pi}\omega(\ell_{jk}-2ih) \quad (18)$$

To satisfy the body condition (10) we have to cancel $u_j$ and $v_j$ (16) in the vicinity of the source $j$ by superimposing on the original source an additional source of strength $\varepsilon_j u_j/2\pi = O(\varepsilon^2)$ and a vertical doublet of strength $-\varepsilon_j^2 v_j/2\pi = O(\varepsilon^3)$. Since we limit the present computations to second order effects, we may disregard at this stage the doublets and consider the sources solely.

The strength of the additional source at $x_j$ is, therefore, given by (17) as

$$\frac{\varepsilon_j u_j}{2\pi} = \frac{\varepsilon_j^2}{4\pi^2}(-2\pi e^{-2h}) + \frac{\varepsilon_j\varepsilon_k}{4\pi^2}\left[-\frac{1}{\ell_{jk}} + \frac{\ell_{ij}}{\ell_{jk}^2+4h^2} - 2\beta(-\ell_{jk},-2h)\right] \quad (19)$$

while at $x_k$ through (18) and (69) we have

$$\frac{\varepsilon_k u_k}{2\pi} = \frac{\varepsilon_k^2}{4\pi^2}(-2\pi e^{-2h}) + \frac{\varepsilon_j\varepsilon_k}{4\pi^2}\left[\frac{1}{\ell_{jk}} - \frac{\ell_{ij}}{\ell_{jk}^2+4h^2} + 2\beta(-\ell_{jk},-2h) - 4\pi e^{-2h}\cos\ell_{jk}\right] \quad (20)$$

The expression of the streamfunction, associated with the two sources, far downstream, is easily found from (14), (19) and (20) as follows

$$\psi_{jk}^{IIb}(x,h) = \operatorname{Im} e^{-ix}\{(\varepsilon_j^2 e^{ix_j} + \varepsilon_k^2 e^{ix_k})iB^b + \varepsilon_j\varepsilon_k[(e^{ix_j} - e^{ix_k})iE_{jk}^b$$

$$+ e^{ix_k}\, iI_{jk}^b]\} \quad (21)$$

where the superscript b stands for body corrections and the coefficients in (21) have the following expressions

$$B^b = 2e^{-3h} \tag{22}$$

$$E^b_{jk} = \frac{e^{-h}}{\pi}\left[\frac{4h^2}{\ell_{jk}(\ell^2_{jk}+4h^2)} + 2\beta(-\ell_{jk},-2h)\right] \tag{23}$$

$$I^b_{jk} = 4e^{-3h}\cos \ell_{jk} \tag{24}$$

$E^b_{jk}$ represents the interaction between the local terms of the two sources and tends to zero like $1/\ell_{jk}$, while $I^b_{jk}$ represents the interaction between the free waves trailing behind source $j$ and the disturbance of source $k$.

3. <u>Second order free-surface correction</u>

The free waves related to the free-surface correction are generated by the linearized pressure $p^{II}(x)$ of Eq. (6). In the case of a pair of sources $p^{II}$ can be written as follows

$$p^{II}_{jk}(x) = \frac{\varepsilon^2_j}{4\pi^2}\tilde{p}_{jj} + \frac{\varepsilon^2_k}{4\pi^2}\tilde{p}_{kk} + \frac{\varepsilon_j\varepsilon_k}{4\pi^2}\tilde{p}_{jk} \tag{25}$$

the different terms resulting from the substitution of the real and imaginary parts of $w^I_{jk}$ and $w^I_{jk,z}$ (12) into (16). Carrying out the substitution we obtain, for instance,

$$\tilde{p}_{jk}(x;x_j,x_k,h) = 4[-\alpha_j\alpha_k-\beta_j\beta_k-\gamma_j\gamma_k+\gamma_j\alpha_k+\gamma_k\alpha_j+\delta_j\beta_k+\delta_k\beta_j+\chi_j\beta_k+\chi_k\beta_j] \tag{26}$$

where, for brevity, the following notations have been used

$$\alpha_\ell = \alpha(x-x_\ell,-h) \qquad \beta_\ell = \beta(x-x_\ell,-h) \qquad \gamma_\ell = \frac{h}{(x-x_\ell)^2+h^2}$$

$$\delta_\ell = \frac{2h(x-x_\ell)}{[(x-x_\ell)^2+h^2]^2} \qquad \chi_\ell = \frac{x-x_\ell}{(x-x_\ell)^2+h^2} \qquad (\ell=j,k) \tag{27}$$

$\tilde{p}_{jk}$ is the pressure resulting from the nonlinear interaction between the velocity components induced by the two sources, while $\tilde{p}_{jj}$ (or $\tilde{p}_{kk}$) stems from the velocity of the same source j (or k). It is readily seen that

$$\begin{aligned}\tilde{p}_{jj}(x;x_j,h) &= \lim_{x_k \to x_j} \frac{1}{2}\tilde{p}_{jk} \\ \tilde{p}_{kk}(x;x_k,h) &= \lim_{x_j \to x_k} \frac{1}{2}\tilde{p}_{jk}\end{aligned} \tag{28}$$

It turns out, therefore, that it is sufficient to determine the free waves generated by the pressure distribution $\tilde{p}_{jk}$ and to take the limits afterwards.

The complex potential representing the flow generated by $\tilde{p}_{jk}$ is (Wehausen and Laitone, 1960)

$$\tilde{f}_{jk} = -\frac{i}{\pi}\int_{-\infty}^{\infty} \tilde{p}_{jk}(s;x_j,x_k,h)\ \omega(z-s-ih)ds \tag{29}$$

If $\tilde{p}_{jk}$, as a function of $s$, is absolutely integrable in the infinite interval, $\omega$ may be expanded for large $x$ under the integral sign of (29). $\tilde{p}_{jk}$ is not integrable as it stays because of the first two terms of (26). We are going to integrate these terms first to eliminate this difficulty. By using integration by parts and (66), (67) we obtain

$$\mathrm{Im}\frac{4i}{\pi}\int_{-\infty}^{\infty}[\alpha_j(s;x_j,h)\alpha_k(s;x_k,h) + \beta_j(s;x_j,h)\beta_k(s;x_k,h)]\omega(x-s)ds =$$

$$= -16\pi^2 e^{-2h}\cos\ell_{jk} - \mathrm{Im}\frac{i}{\pi}\int_{-\infty}^{\infty} 4i(\beta_j\gamma_k+\gamma_j\beta_k+\alpha_k\chi_j+\alpha_k\chi_j)\omega(x-s)ds \tag{30}$$

After this first integration the pressure is integrable and by expanding $\omega(x-s)$ for $x \to \infty$ (Eq. 65) we obtain for large x

$$\tilde{\psi}_{jk} = \operatorname{Im} \tilde{f}_{jk} = -16\pi^2 e^{-2h} \cos \ell_{jk} + \operatorname{Im} 8e^{-ix} \int_{-\infty}^{\infty} [i(\beta_j \gamma_k + \gamma_j \beta_k + \alpha_j \chi_k + \alpha_k \chi_j)$$

$$- \gamma_j \gamma_k + \gamma_j \alpha_k + \gamma_k \alpha_j + \delta_j \beta_k + \delta_k \beta_j + \chi_j \beta_k + \chi_k \beta_j] e^{is} ds \qquad (x \to \infty) \qquad (31)$$

The integration of the various terms of (31) may be carried out by using the representations of $\alpha_\ell$, $\beta_\ell$ given in Appendix (63). We have, for instance,

$$\int_{-\infty}^{\infty} (\alpha_j \gamma_k + \beta_j \chi_k) e^{is} ds = \int_{-\infty}^{\infty} [\operatorname{Im} \int_0^{\infty} \frac{e^{-i\rho}}{(s-\rho-x_j-ih)(s-x_k-ih)} d\rho] e^{is} ds \qquad (32)$$

$$\int_{-\infty}^{\infty} i(\beta_j \gamma_k + \alpha_j \chi_k) e^{is} ds = i \int_{-\infty}^{\infty} [\operatorname{Re} \int_0^{\infty} \frac{e^{-i\rho}}{(s-\rho-x_j-ih)(s-x_k+ih)} d\rho] e^{is} ds \qquad (33)$$

and by using also (67)

$$\int_{-\infty}^{\infty} \beta_j \delta_k e^{is} ds = \int_{-\infty}^{\infty} \gamma_k \frac{d}{ds} (\beta_j e^{is}) ds = \int_{-\infty}^{\infty} e^{is} \int_0^{\infty} \frac{e^{i\rho}}{(s-\rho-x_j+ih)[(s-x_k)^2+h^2]} d\rho ds +$$

$$+ \frac{1}{h} \int_{-\infty}^{\infty} \gamma_j \gamma_k e^{is} ds \qquad (34)$$

Using the residue and the semiresidue theorems and the definitions of $\alpha$ and $\beta$ (63) we arrive after integration in (34) to the final expression of $\tilde{\psi}_{jk}$

$$\tilde{\psi}_{jk}(x,h) = 8\pi e^{-h} \operatorname{Im} e^{-ix} \{ (e^{ix_j} + e^{ix_k}) [\ln \sqrt{1 + \frac{4h^2}{\ell_{jk}^2}} + \alpha(-\ell_{jk}, 0) -$$

$$- 2\alpha(-\ell_{jk}, -2h) + \frac{h}{\ell_{jk}^2 + 4h^2}] + (e^{ix_j} - e^{ix_k}) i [-\tan^{-1} \frac{2h}{\ell_{jk}} +$$

$$+ \beta(-\ell_{jk}, 0) + 2\beta(-\ell_{jk}, -2h) + \frac{2h^2}{\ell_{jk}(\ell_{jk}^2 + 4h^2)}] +$$

$$+ 2\pi i e^{ix_k} (e^{-i\ell_{jk}} + 2e^{-2h} e^{i\ell_{jk}}) \} - 16\pi^2 e^{-2h} \cos \ell_{jk} \qquad (x \to \infty) \qquad (35)$$

$\tilde{\psi}_{jj}$ and $\tilde{\psi}_{kk}$ are obtained from $\tilde{\psi}_{jk}$ by taking the limit $\ell_{jk} \to 0$ (Eqs. (28)). Carrying out the computations and using the properties of the functions $\alpha$ and $\beta$ (69), yields the following final expression for the streamfunction far downstream

$$\psi_{jk}^{IIs} = \operatorname{Im} e^{-ix}\{(\varepsilon_j^2 e^{ix_j} + \varepsilon_k^2 e^{ix_k})(A^s + iB^s) + \underset{j<k}{\varepsilon_j \varepsilon_k}[(e^{ix_j} + e^{ix_k})C_{ij}^s$$

$$+ (e^{ix_j} - e^{ix_k})iE_{jk}^s + e^{ix_k} i(I_{jk}^s + iK_{jk}^s)]\} + (\varepsilon_j^2 + \varepsilon_k^2)G^s + \varepsilon_j \varepsilon_k G^s \cos \ell_{jk}$$

$$(x \to \infty) \qquad (36)$$

where

$$A^s = \frac{2}{\pi} e^{-h}[\ln 2h + \frac{1}{4} + C - 2e^{-2h}\,\overline{Ei}(2h) + \frac{1}{4h}] \qquad (37)$$

$$B^s = 2e^{-h}(1-2e^{-2h}) \qquad (38)$$

$$C_{jk}^s = \frac{2}{\pi} e^{-h}[\frac{1}{2} \ln(1 + \frac{4h^2}{\ell_{jk}^2}) + \alpha(-\ell_{jk}, 0) - 2\alpha(-\ell_{jk}, -2h) + \frac{h}{\ell_{jk}^2 + 4h^2}] \qquad (39)$$

$$E_{jk}^s = \frac{2}{\pi} e^{-h}[-\tan^{-1} \frac{2h}{\ell_{jk}} + \beta(-\ell_{jk}, 0) + 2\beta(-\ell_{jk}, -2h) + \frac{2h^2}{\ell_{jk}(\ell_{jk}^2 + 4h^2)}] \qquad (40)$$

$$I_{jk}^s = 4e^{-h}(1+2e^{-2h})\cos \ell_{jk} \qquad (41)$$

$$K_{jk}^s = 4e^{-h}(-1+2e^{-2h})\sin \ell_{jk} \qquad (42)$$

$$G^s = -2e^{-2h} \qquad (43)$$

The superscript $s$ stands for free-surface corrections, $C = 0.5772 \ldots$ in (37) is the Euler constant and $\overline{Ei}(2h)$ in (37) is the exponential integral.

The first term in (35), associated with $A^s$ and $B^s$, represents the free waves generated by the interaction between the different components of the velocity of the same source, while the remaining term, in $\varepsilon_j \varepsilon_k$, is related to the interaction between the two sources. The last term, in $G^S$, is the nonperiodic D.C. term which does not contribute to the wave resistance.

4. Asymptotic expansions for the free waves amplitude

The amplitude and the phase of the free waves depend entirely on $\varepsilon_j$, $\varepsilon_k$, $x_j$, $x_k$ and the coefficients of (36)-(42). Since we have found closed analytical expressions for these coefficients it is easy to study their behavior for large and small Froude numbers.

*Small Froude number.* We consider the limit $h = h'g/U'^2 \to \infty$ for a fixed ratio $h/\ell_{jk} = h'/\ell'_{jk}$, i.e. we let also $\ell_{jk} = \ell'_{jk}g/U'^2 \to \infty$.

Under this limit the amplitude of the first order waves (15) $a^I$ is $O(\varepsilon e^{-h})$. The amplitude of the waves resulting from the body correction (22-24) $a^{IIb}$ is $O(\varepsilon^2 e^{-h}/\ell_{jk})$ or $O(\varepsilon^2 e^{-3h})$. Finally, the largest possible amplitude of the waves associated with the free-surface correction $a^{IIs}$ stems from $A^S$ (37) and is $O(\varepsilon^2 e^{-h} \ln h)$. Hence, $a^{IIb}/a^I = O(\varepsilon/\ell_{jk})$ or $O(\varepsilon e^{-2h})$ and $a^{IIs}/a^I = O(\varepsilon \ln h)$.

Since $\varepsilon$ is the inverse of the thickness Froude number, the singular behavier of the free waves for $h \to \infty$ found in previous works (Salvesen 1969, Dagan 1971) is validated. Moreover, the order of magnitude of $a^{IIs}/a^I$ for $U' \to 0$ is exactly found $a^{IIs}/a^I \sim \ln U'/U'^2$.

This point is discussed in more detail in the analysis of the wave resistance (Section IV).

*Large Froude number.* With $h = \frac{h'g}{U'^2} \to 0$, $\ell_{jk} \to 0$ and $\frac{h}{\ell_{jk}} = O(1)$ we obtain from (15), (23) and (39) or (40) for the largest possible amplitudes

$$a^I = O(\varepsilon), \quad a^{IIb} = O(\varepsilon^2/\ell_{jk}) \quad \text{and} \quad a^{IIs} = O(\varepsilon^2/\ell_{jk}) .$$

Hence the ratio $a^{II}/a^I$ is of order $\varepsilon/\ell_{jk} = T'/\ell'_{jk}$ at most, the latter being a fixed (generally small) number for a given body.

5. Free waves behind an arbitrary source distribution

For an arbitrary distribution of $n$ sources of strength $\varepsilon_\ell$ located at $x = x_\ell$, $y = 0$ $(\ell=1,2,\ldots,n)$ we obtain from (15), (21) and (36)

$$\psi^{I}(x,h) = \mathrm{Im}[(-2ie^{-h})e^{-ix}\sum_{j=1}^{n}\varepsilon_j e^{ix_j}] \tag{44}$$

$$\begin{aligned}\psi^{II}(x,h) = \psi^{IIb} + \psi^{IIs} = \mathrm{Im}\; e^{-ix}\{ &\sum_{j=1}^{n}\varepsilon_j^2 e^{ix_j}[A^s+i(B^b+B^s)] + \\ &+ \sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\varepsilon_j\varepsilon_k[(e^{ix_j}+e^{ix_k})C^s_{jk} + (e^{ix_j}-e^{ix_k})i(E^b_{jk}+E^s_{jk}) + \\ &+ e^{ix_k}i(I^b_{jk}+I^s_{jk}+iK^s_{jk})]\} + G^s(\sum_{j=1}^{n}\varepsilon_j^2 + 2\sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\varepsilon_j\varepsilon_k\cos\ell_{jk})\end{aligned} \tag{45}$$

where the coefficients of (45) are given in (22-24) and (37-43).

## IV. THE WAVE RESISTANCE

1. General formulation

The wave resistance may be determined, for instance, from a momentum balance between two vertical sections far ahead and behind the body.

With the free waves profile (8,9) given by

$$\eta^{I}(x) = -\psi^{I}(x,h) = -(\psi^{I}_{\cos}\cos x + \psi^{I}_{\sin}\sin x) \tag{46}$$

$$\eta^{II}(x) = \eta^{I^2} - \psi^{II} = \eta^{I^2} - (\psi^{II}_{\cos}\cos x + \psi^{II}_{\sin}\sin x) - \psi^{II}_{const} \tag{47}$$

the wave resistance at second order (Salvesen, 1969) is given by

$$D = D^I + D^{II} + O(\varepsilon^4) = \frac{\psi_{\cos}^{I^2} + \psi_{\sin}^{I^2}}{4} + \frac{1}{2}\,(\psi_{\cos}^{I}\psi_{\cos}^{II} + \psi_{\sin}^{I}\psi_{\sin}^{II}) + O(\varepsilon^4) \quad (48)$$

In (46), (47) and (48) $\psi_{\cos}^{I}$, $\psi_{\sin}^{I}$, $\psi_{\cos}^{II}$ and $\psi_{\sin}^{II}$ represent the periodical sin x and cos x terms of (44) and (45), respectively, while $\psi_{const}^{II}$ stands for the last term of (45). $\psi_{const}^{II}$ is exactly canceled in (47) by the constant part of $\eta^{I^2}$ such that the average free surface elevation is equal to h for $x \to \infty$. The dimensionless drag in (48) is defined as $D = D'g/\rho U'^4$. Using Eqs. (44) and (45) and substituting in (48), we obtain after some manipulations

$$D^I = e^{-2h} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j \varepsilon_k \cos \ell_{jk} \quad (49)$$

$$D^{II} = D^{IIb} + D^{IIs} = -e^{-h}\Big( \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_j \varepsilon_k^2 [A^s \sin\ell_{jk} + (B^b + B^s)\cos\ell_{jk}]$$

$$+ \sum_{m=1}^{n} \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \varepsilon_m \varepsilon_j \varepsilon_k [C_{jk}^s (\sin\ell_{mj} + \sin\ell_{mk}) + (E_{jk}^b + E_{jk}^s)(\cos\ell_{mj} - \cos\ell_{mk})$$

$$+ (I_{jk}^b + I_{jk}^s)\cos\ell_{mk} - K_{jk}^s \sin\ell_{mk}]\Big) \quad (50)$$

Eqs. (49) and (50) give in a closed form the first and second order wave resistance, respectively, for an arbitrary source distribution. $D^{IIb}$ and $D^{IIs}$, the nonlinear body and free-surface contributions, may be computed separately from (50) by selecting the appropriate coefficients.

We are going now to discuss the application of (49) and (50) to a few simple, but instructive, particular cases.

2. The isolated source

The simplest case conceivable is that of an isolated source beneath a free-surface (Fig. 2a) representing, at second order, a body of semi-infinite length with a parabolical blunt leading edge.

The wave resistance components, obtained from (49), (50) with $n=1$, have the simple expressions

$$
\begin{aligned}
D^{I} &= \varepsilon^2 e^{-2h} \\
D^{IIb} &= \varepsilon^3(-2e^{-4h}) \\
D^{IIs} &= \varepsilon^3[-2e^{-2h}(1+2e^{-2h})]
\end{aligned}
\tag{51}
$$

The wave resistance of (51) is given in Fig. 2c by using, however, the more common drag coefficient

$$C_D = D'/0.5\rho U'^2 T' \quad \text{and the parameter} \quad T'/h' = t.$$

Hence, with

$$C_D = tC_D^{I} + t^2 C_D^{II}$$

we have

$$
\begin{aligned}
C_D^{I} &= 2he^{-2h} \\
C_D^{IIb} &= -4h^2 e^{-4h} \\
C_D^{IIs} &= -4e^{-2h}\, h^2(1+2e^{-2h})
\end{aligned}
\tag{52}
$$

The first order coefficient $C_D^{I}$ as function of $U'/\sqrt{gh'} = 1/\sqrt{h}$ has the well-known shape with a maximum at $U'^2/gh' = 2$. The second order resistance coefficients are negative, i.e. they always diminish the values of the first order drag. To illustrate the relative magnitude of the second order effect, the ratio $C_D^{II}/C_D^{I}$, which does not depend on $t$, has also been represented in Fig. 2. For $U'/\sqrt{gh'} \to \infty$ this ratio tends to zero like $-8h$ and, therefore $t^2C_D^{II}/tC_D^{I} \to -8ht = -8\epsilon$. Hence, $1/\epsilon = U'^2/gT'$ has to be much larger than 8 in order to ensure small nonlinear effect at high Froude numbers. For $U'/\sqrt{gh'} \to 0$, $C_D^{II}/C_D^{I} \to -2h$ i.e. $t^2C_D^{II}/tC_D^{I} \to -2\epsilon$, and the nonlinear free-surface correction becomes much larger than the first order wave resistance because of the free-surface effect. In the intermediate range $C_D^{II}/C_D^{I}$ grows monotonically (Fig. 2).

Finally, to estimate the relative importance of the two nonlinear effects, the ratio $C_D^{IIs}/C_D^{IIb} = D^{IIs}/D_b^{II} = D'^{IIs}/D'^{IIb}$ is also represented in Fig. 2c. For $U'/\sqrt{gh'} \to \infty$ this ratio tends to 3. As $U'/\sqrt{gh'}$ decreases $D^{IIs}/D_b^{II}$ tends to infinity like $e^{2h}$. Hence, in the present case, the free-surface correction is always much larger than the body correction.

## 3. The influence of the fineness of the leading edge shape

To study the influence of the shape of the leading edge we have considered a semi-infinite body generated by ten sources of equal strength and at equal spacings, i.e. $\epsilon_j = \epsilon_k = \epsilon/10$ and $\ell_{jk} = (k-j)1/9$ (Fig. 2b).

This distribution represents quite accurately a wedge shaped nose. It turns out, now, that the general drag formulae (49) and (50) simplify in this case and become

$$D^{I} = 10^{-2}\epsilon^2 e^{-2h} \sum_{j=1}^{10}\sum_{k=1}^{10} \cos\ell_{jk} \tag{53}$$

$$D^{IIb} = -10^{-3}\epsilon^3 e^{-h}\Big[B^b \sum_{j=1}^{10}\sum_{k=1}^{10} \cos\ell_{jk} + \sum_{m=1}^{10}\sum_{j=1}^{9}\sum_{k=j+1}^{10} I_{jk}^{b}\cos\ell_{mk}\Big] =$$

$$= -10^{-3}\epsilon^3 e^{-h}(2e^{-4h})\Big[\sum_{j=1}^{10}\sum_{k=1}^{10}\cos\ell_{jk} + 2\sum_{m=1}^{10}\sum_{j=1}^{9}\sum_{k=j+1}^{10} \cos\ell_{mk}\cos\ell_{jk}\Big] \tag{54}$$

$$D^{IIs} = -10^{-3}\epsilon^3 e^{-h}[B^s \sum_{j=1}^{10}\sum_{k=1}^{10}\cos\ell_{jk} + \sum_{m=1}^{10}\sum_{j=1}^{9}\sum_{k=j+1}^{10}(I^s_{jk}\cos\ell_{mk} - K^s_{jk}\sin\ell_{mk})] =$$

$$= -10^{-3}\epsilon^3[2e^{-2h}(1+2e^{-2h})\sum_{j=1}^{10}\sum_{k=1}^{10}\cos\ell_{jk} + 2e^{-2h}(1+2e^{-2h}) \cdot \tag{55}$$

$$\cdot \sum_{m=1}^{10}\sum_{j=1}^{9}\sum_{k=j+1}^{10}\cos\ell_{mk}\cos\ell_{jk} - 4e^{-2h}(1-2e^{-2h})\sum_{m=1}^{10}\sum_{j=1}^{9}\sum_{k=j+1}^{10}\sin\ell_{mk}\sin\ell_{jk}]$$

$D^I/\epsilon^2$, $D^{IIb}/\epsilon^3$ and $D^{IIs}/\epsilon^3$ have been computed as function of $h$ for different ratios between the fore body length and the submergence depth $(L'/h')$.

In Fig. 2c we give the results, again in terms of $C_D = D'/0.5\rho U'^2 T =$ $= tC_D^I + t^2 C_D^{II}$ and $t = T'/h'$ for $L'/h' = 9$ and $L'/h' = 4.5$. The first order resistance coefficient $C_D^I$ drops markedly as the ratio $L'/h'$ increases. For $U'/\sqrt{gh'} > 3$ the curves become very close, because the ratio between the length of the free waves $\lambda' = 2\pi U'^2/g$ and the length of the fore body $L'$ is larger than $2\pi$ for the largest $L'$ considered. At smaller Froude numbers, however, the interference effects are manifest.

A more interesting point is that related to the influence of the fineness on the ratio $C_D^{II}/C_D^I$ (Fig. 3). Excepting for large $(U'/\sqrt{gh'} > 3)$ or small $(U'/\sqrt{gh'} < 0.7)$ Froude numbers, the relative nonlinear effect is smaller for fine shapes, as much as half of that of the blunt shape (for $L'/h' = 9$). For the latter case $C_D^{II}/C_D^I$ is almost constant for $3 > U'/gh' > 1.7$ and approximately equal to 0.75.

Finally, the fineness has little influence on the ratio $D^{IIs}/D^{IIb}$ (Fig. 2c) which remains large, irrespective of the shape of the leading edge.

4. Interference effects: the source-sink body

We consider now a body generated by a source and a sink (Fig. 3a), which is the simplest case of a closed body.

Using again the general expression of the drag ((49),(50)) with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = -\varepsilon$ and $\ell_{12} = L$ we obtain in this case

$$D^{I} = \varepsilon^2 2e^{-2h}(1-\cos L) \tag{56}$$

$$D^{IIb} = -\varepsilon^3 e^{-h}[-2H^{b}_{12}(1-\cos L) + I^{b}_{12}(1-\cos L)] \tag{57}$$

$$D^{IIs} = -\varepsilon^3 e^{-4}[2A^{s}\sin L - 2C^{s}_{12}\sin L - 2H^{s}_{12}(1-\cos L) + I^{s}_{12}(1-\cos L) + K^{s}_{12}\sin L] \tag{58}$$

the different coefficients being given by (22)-(24) and (37)-(42). We have selected an example with $L' = 20h'$ consistent with considering elongated bodies resembling ships. In this ca e we can use the asymptotic expressions of $\alpha$ and $\beta$ (65) which appear in $C^{s}_{12}$ and $I^{s}_{12}$ (39),(40) to obtain

$$C^{s}_{12} = \frac{2}{\pi} e^{-h} \frac{1-3h+2h^2}{L^2} + O(L^{-3})$$

$$I^{s}_{12} = \frac{3-2h}{L} + O(L^{-2}) \tag{59}$$

these coefficients being related to the interaction between the local disturbances. Only terms of order $L^{-1} = (20h)^{-1}$ or larger have been retained in (58). To plot the different drag component, we have used the more familiar dimensionless parameters $C_L = D'/\rho U'^2 L'$ and $\varepsilon_L = T'/L'$. With this representation we have

$$C_L = \varepsilon_L^2 C_L^{I} + \varepsilon_L^3 C_L^{II} \tag{60}$$

where

$$C_L^I = \frac{L'}{h'} hD^I = 20hD^I$$

$$C_L^{II} = \left(\frac{L'}{h'}\right)^2 h^2 D^{II} = 400h^2 D^{II} \tag{61}$$

In Fig. 3b $C_L^I$, $C_L^{II}$ and $C_L^{IIb}$ are represented separately as functions of $U'/\sqrt{gL'}$, all these functions having an oscillatory character. Like in the previous cases, the body correction is smaller than the free-surface second order correction. For sufficiently high Froude numbers ($U'/\sqrt{gL'} > 0.3$) $C_L^{IIb}$ is roughly $0.2 \div 0.3 C_L^{II}$, while for $U'/\sqrt{gL'} < 0.3$ its relative magnitude becomes even smaller. Moreover, for the ratio $L'/h' = 20$ considered here, the local terms in (57) and (58), i.e. the terms in $E_{12}^b$, $A^s$, $C_{12}^s$, $E_{12}^s$, are small compared with the other terms and, for the sake of the analysis, $D^{IIb}$ and $D^{IIs}$ may be approximated for $0.15 < U'/\sqrt{gL'} < 0.45$ by

$$D^{IIb} \simeq \varepsilon^3 e^{-h} I_{12}^b \cos L(1-\cos L) = -\varepsilon^3 4e^{-4h} \cos L(1-\cos L)$$

$$D^{IIs} \simeq -\varepsilon^3 e^{-2h} I_{12}^s (1-\cos L) = -\varepsilon^3 4e^{-2h}(1+2e^{-2h}) \cos L(1-\cos L) \tag{62}$$

Hence, $D^I$ (56) oscillates like $1-\cos L$ while $D^{IIs}$ and $D^{IIb}$ behave like $\cos L(1-\cos L)$. As a result, the second order correction tends to sharpen the peaks of the resistance curve and to make the hollows wider. At the peak of the resistance curve for $U'/\sqrt{gL'} = 0.32$ the ratio $D^{II}/D^I$ is of order $40(T'/L')$ and this ratio grows as the Froude number decreases. Like in the case of the semi-infinite body, the free-surface second order effect becomes unboundedly larger than the first order drag for $U'/\sqrt{gL'} \to 0$, the dominant term in (58) being $A^s \sim e^{-2h} \ln h$.

## V. DISCUSSION OF RESULTS AND CONCLUSIONS

Although an useful closed solution for the nonlinear wave resistance in two-dimensions has been obtained, the main purpose of the present work is to draw qualitative conclusions on nonlinear ship wave resistance. Two features of the results, summarized in Figs. 2 and 3, seem to be general: (i) the free-surface second order contribution to the drag is larger than that of the body correction, and (ii) at small Froude numbers the nonlinear effects become very large and the basic expansion of the velocity field is singular.

The results depicted in Fig. 2 seem at first glance of limited interest, because they apply to open bodies. But there are indications that due to viscous effects the stern waves of ships are smaller than those predicted by the sink distribution representing the geometry in the linearized theory. Such an assumption is tantamount to considering an open body so far as wave resistance is concerned. The results of Fig. 2 may, therefore, have some bearing on the part of the wave resistance associated with the bow waves. It suggests that: (i) the nonlinear wave resistance is smaller than the first order drag, and (ii) that the second order effect is less important for fine shapes than for full forms of the fore body for moderate Froude numbers ($U'/\sqrt{gL'} \approx 0.5$, where $L'$ is the fore body length).

The interference effects between the leading and trailing edges are reflected by the results of Fig. 3. Obviously, these effects will be largely reduced if the trailing edge is assumed to be of a fine shape or open (to account for wake, for instance). At any rate, the present computations indicate that the resistance peaks are augmented by second order effects while the hollows become wider.

The method used in the present work may be easily extended to account for vorticity in two dimensions. It establishes a pattern which may be useful in attacking the problem of three-dimensional source generated bodies.

## VI. APPENDIX - The Properties of the Function $\omega(\zeta)$.

The function $\omega(\zeta) = \alpha(\xi,\eta) + i\beta(\xi,\eta)$ is defined by (13) as

$$\omega(\zeta) = e^{-i\zeta}\overline{Ei}(i\zeta) = e^{-i\zeta}\int_{-\infty}^{\zeta} \frac{e^{i\lambda}}{\lambda}\, d\lambda$$

in the $\lambda$ complex plane (Fig. 4). Another expression, obtained by the change of variables $\rho = -\lambda + \zeta$, is

$$\omega(\zeta) = -\int_{0}^{\infty} \frac{e^{-i\rho}}{\rho - \zeta} \tag{63}$$

with the $\rho$ plane depicted in Fig. 4.

$\omega$ has the following asymptotic expansions:

$$\omega = e^{-i\zeta}(i\pi + C + \ln\zeta + \sum_{m=1}^{\infty} \frac{\zeta^m}{m\cdot m!}) \qquad (\zeta \to 0) \tag{64}$$

and

$$\omega = \sum_{m=1}^{\infty} \frac{(n-1)!}{(i\zeta)^n} + \omega_\infty(\zeta) \qquad (\zeta \to \infty) \tag{65}$$

where $\omega_\infty(\zeta) = 2\pi i e^{-i\zeta}$ for $\mathrm{Re}\,\zeta \to \infty$ and $\omega_\infty = 0$ for $\mathrm{Re}\,\zeta \to -\infty$. The series of (65) is divergent.

Herewith a few relationships used in the present work:

(i) $$\frac{d\omega}{d\zeta} = -i\omega + \frac{1}{\zeta} \tag{66}$$

and particularly

$$\begin{aligned} \alpha(\xi,\eta) &= -\frac{\partial\beta}{\partial\xi} - \frac{\eta}{\xi^2 + \eta^2} \\ \beta(\xi,\eta) &= \frac{\partial\alpha}{\partial\xi} - \frac{\xi}{\xi^2 + \eta^2} \quad ; \end{aligned} \tag{67}$$

(ii) $$\omega(\zeta) = 2\pi i e^{-i\zeta} + \omega(-\xi + i\eta) \tag{68}$$

which leads to

$$\begin{aligned} \alpha(\xi,\eta) &= 2\pi e^{\eta}\sin\xi + \alpha(-\xi,\eta) \\ \beta(\xi,\eta) &= 2\pi e^{\eta}\cos\xi - \beta(-\xi,\eta) . \end{aligned} \tag{69}$$

To compute $\alpha$ and $\beta$ one may use the series of (64) for $|\zeta| < 10$ or series (65) for $|\zeta| > 10$. In the latter case the series has to be truncated at $m = N$ where $N \sim |\zeta|$ .

$\omega(\zeta)$ is related to the function $E_1(\zeta)$ tabulated by Abramowitz and Stegun (1964) as follows:

$$\omega(\zeta) = e^{-i\zeta}\,[2\pi i - E_1(-i\zeta)] \quad . \tag{70}$$

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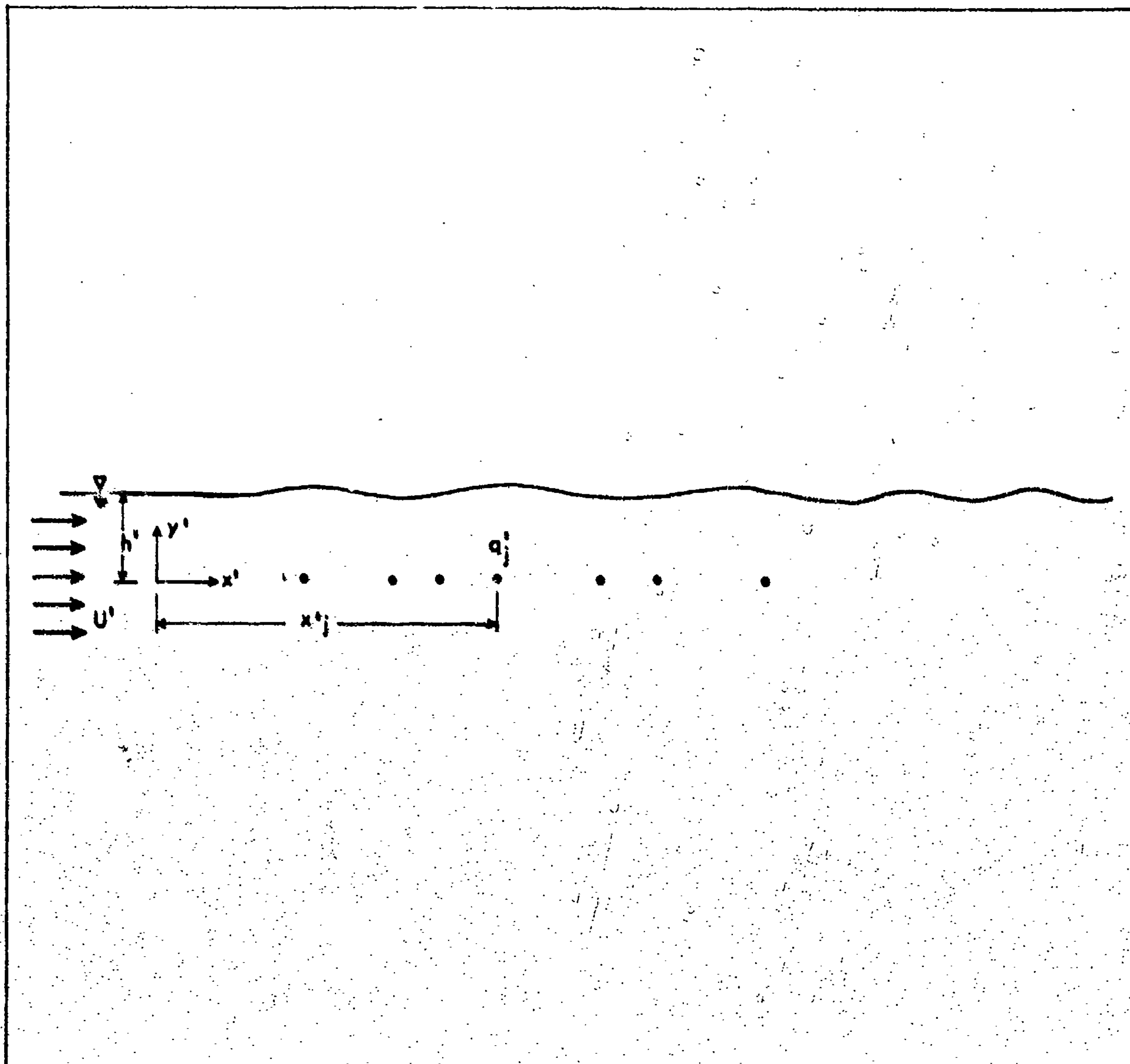


FIGURE 1 - A DISTRIBUTION OF SOURCES.

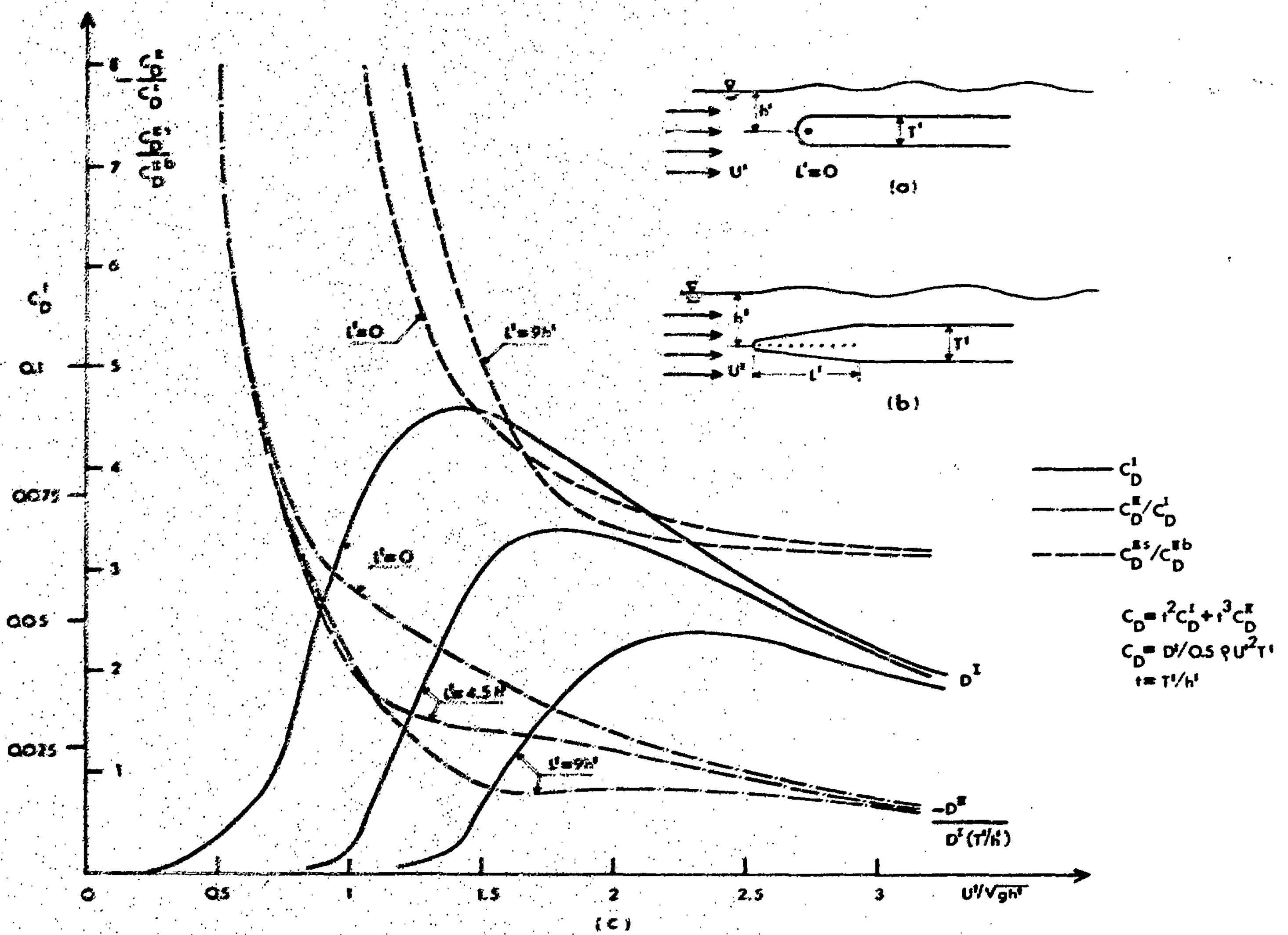


FIGURE 2 - WAVE RESISTANCE OF A BODY OF SEMI-INFINITE LENGTH: (a) A SOURCE GENERATED BODY, (b) WEDGE SHAPE LEADING EDGE, AND (c) WAVE RESISTANCE CURVES.

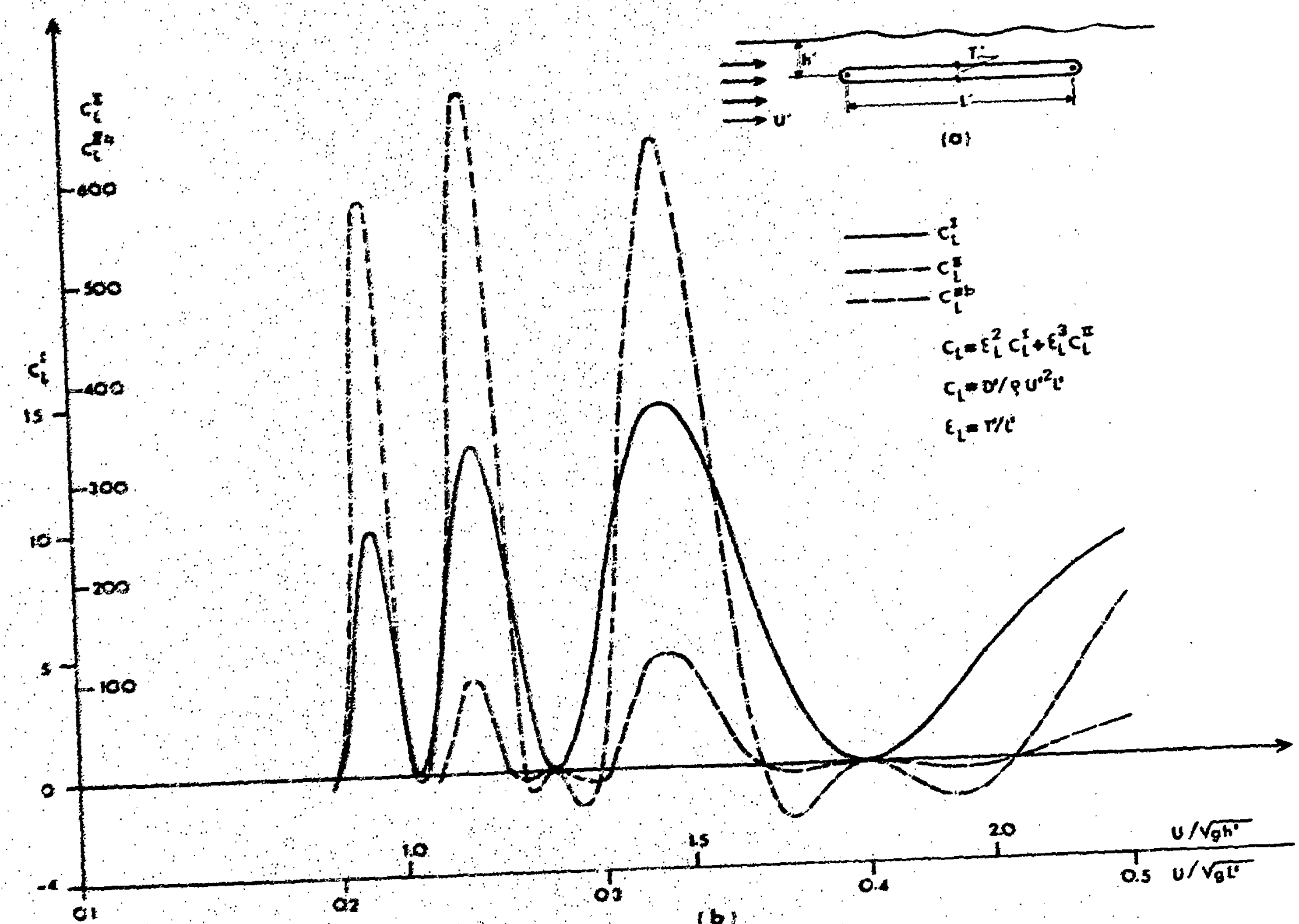


FIGURE 3 - WAVE RESISTANCE OF A SOURCE-SINK BODY: (a) THE BODY SHAPE AND (b) WAVE RESISTANCE CURVES.

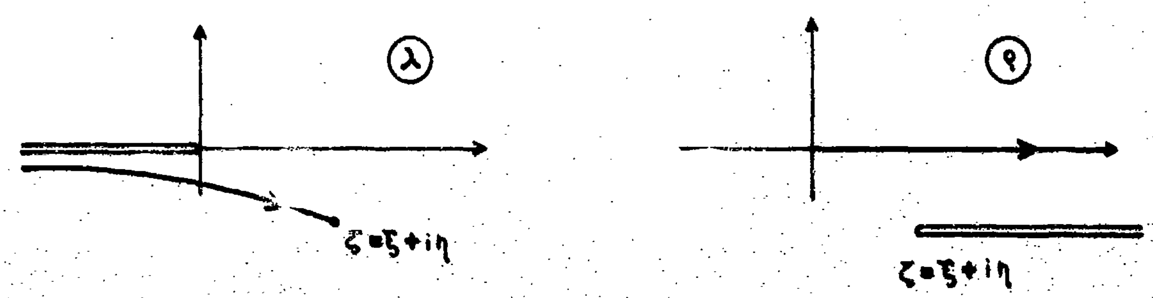


FIGURE 4 - THE λ AND ρ PLANES.